e content for students of patliputra university

B. Sc. (Honrs) Part 2paper 4

Subject:Mathematics

Topic:Products of Four vectors

Scalar product of four vectors

If $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ be any four vectors, the product of the type $(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d})$, $(\vec{a} \times \vec{c}) \cdot (\vec{b} \times \vec{d})$, etc. are called *scalar product of four vectors.*

By definition of cross product of two vectors, we find that $\vec{a} \times \vec{b}$ and $\vec{c} \times \vec{d}$ are vector quantities.

Let $\vec{a} \times \vec{b} = \vec{p}$ and $\vec{c} \times \vec{d} = \vec{q}$.

Then $(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) = \vec{p} \cdot \vec{q}$, which is, by definition of scalar product of two vectors, a scalar quantity. But it is the product of four vectors. Hence $(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d})$ is called scalar product of four vectors.

Q To find the expansion of $(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d})$.
Or, to prove that

$$(\vec{a} \times \vec{b}).(\vec{c} \times \vec{d}) = (\vec{a} \cdot \vec{c})(\vec{b} \cdot \vec{d}) - (\vec{a} \cdot \vec{d})(\vec{b} \cdot \vec{c})$$

$$= \begin{vmatrix} \vec{a} \cdot \vec{c} & \vec{a} \cdot \vec{d} \\ \vec{b} \cdot \vec{c} & \vec{b} \cdot \vec{d} \end{vmatrix}.$$

Proof. Let $\vec{a} \times \vec{b} = \vec{m}$.

Then $(\vec{a} \times \vec{b}).(\vec{c} \times \vec{d}) = \vec{m} \cdot (\vec{c} \times \vec{d})$

$=(\vec{m} \times \vec{c}) \cdot \vec{d}$,

[as in a scalar triple product the position of dot and cross can be interchanged provided the cyclic order is maintained]

$=\{(\vec{a} \times \vec{b}) \times \vec{c}\} \cdot \vec{d}$

$=\{(\vec{a} \cdot \vec{c})\, \vec{b} - (\vec{b} \cdot \vec{c})\, \vec{a}\} \cdot \vec{d}$

$=(\vec{a} \cdot \vec{c})(\vec{b} \cdot \vec{d})-(\vec{a} \cdot \vec{d})(\vec{b} \cdot \vec{c})$

$$=\begin{vmatrix} \vec{a} \cdot \vec{c} & \vec{a} \cdot \vec{d} \\ \vec{b} \cdot \vec{c} & \vec{b} \cdot \vec{d} \end{vmatrix},$$

vector product of four vectors

Definition. If $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ be any four vectors, the products of the type $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d})$, $(\vec{a} \times \vec{c}) \times (\vec{b} \times \vec{d})$ etc. are called *vector product of four vectors.*

By definition of vector product of two vectors, we find that $\vec{a} \times \vec{b}$ and $\vec{c} \times \vec{d}$ are vector quantities.

Let $\quad \vec{a} \times \vec{b} = \vec{p}$ and $\vec{c} \times \vec{d} = \vec{q}$.

Then $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = \vec{p} \times \vec{q}$, which is again a vector quantity.

Therefore $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d})$ is a vector quantity. But it is the product of four vectors.

Hence $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d})$ is called vector product of four vectors.

(i) To find the expansion of $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d})$.
Or, to prove that

$$(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = [\vec{a}\ \vec{b}\ \vec{d}]\,\vec{c} - [\vec{a}\ \vec{b}\ \vec{c}]\,\vec{d}$$
$$= [\vec{a}\ \vec{c}\ \vec{d}]\,\vec{b} - [\vec{b}\ \vec{c}\ \vec{d}]\,\vec{a}.$$

Proof. (i) Let $\vec{a} \times \vec{b} = \vec{m}$.

Then $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = \vec{m} \times (\vec{c} \times \vec{d})$

$$= (\vec{m} \cdot \vec{d})\,\vec{c} - (\vec{m} \cdot \vec{c})\,\vec{d}$$
$$= \{(\vec{a} \times \vec{b}) \cdot \vec{d}\}\,\vec{c} - \{(\vec{a} \times \vec{b}) \cdot \vec{c}\}\,\vec{d}$$
$$= [\vec{a}\ \vec{b}\ \vec{d}]\,\vec{c} - [\vec{a}\ \vec{b}\ \vec{c}]\,\vec{d}.$$

Again, let $\vec{c} \times \vec{d} = \vec{n}$.

Then $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = (\vec{a} \times \vec{b}) \times \vec{n}$

$$= (\vec{a} \cdot \vec{n})\,\vec{b} - (\vec{b} \cdot \vec{n})\,\vec{a}$$
$$= \{\vec{a} \cdot (\vec{c} \times \vec{d})\}\,\vec{b} - \{\vec{b} \cdot (\vec{c} \times \vec{d})\}\,\vec{a}$$
$$= [\vec{a}\ \vec{c}\ \vec{d}]\,\vec{b} - [\vec{b}\ \vec{c}\ \vec{d}]\,\vec{a}.$$

(ii) From part (i), we have

$$(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = [\vec{a}\ \vec{b}\ \vec{d}]\,\vec{c} - [\vec{a}\ \vec{b}\ \vec{c}]\,\vec{d}$$

and $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = [\vec{a}\ \vec{c}\ \vec{d}]\,\vec{b} - [\vec{b}\ \vec{c}\ \vec{d}]\,\vec{a}$.

$\therefore \quad [\vec{a}\ \vec{b}\ \vec{d}]\,\vec{c} - [\vec{a}\ \vec{b}\ \vec{c}]\,\vec{d} = [\vec{a}\ \vec{c}\ \vec{d}]\,\vec{b} - [\vec{b}\ \vec{c}\ \vec{d}]\,\vec{a}.$

Hence $[\vec{b}\ \vec{c}\ \vec{d}]\,\vec{a} - [\vec{a}\ \vec{c}\ \vec{d}]\,\vec{b} + [\vec{a}\ \vec{b}\ \vec{d}]\,\vec{c} - [\vec{a}\ \vec{b}\ \vec{c}]\,\vec{d} = \vec{0}$.

(iii) From part (ii), we have

$$[\vec{b}\ \vec{c}\ \vec{d}]\,\vec{a} - [\vec{a}\ \vec{c}\ \vec{d}]\,\vec{b} + [\vec{a}\ \vec{b}\ \vec{d}]\,\vec{c} - [\vec{a}\ \vec{b}\ \vec{c}]\,\vec{d} = \vec{0}$$

or $[\vec{a}\ \vec{b}\ \vec{c}]\,\vec{d} = [\vec{b}\ \vec{c}\ \vec{d}]\,\vec{a} + [\vec{c}\ \vec{a}\ \vec{d}]\,\vec{b} + [\vec{a}\ \vec{b}\ \vec{d}]\,\vec{c}$.

Hence $\vec{d} = \dfrac{[\vec{b}\ \vec{c}\ \vec{d}]\,\vec{a} + [\vec{c}\ \vec{a}\ \vec{d}]\,\vec{b} + [\vec{a}\ \vec{b}\ \vec{d}]\,\vec{c}}{[\vec{a}\ \vec{b}\ \vec{c}]}$,

as $[\vec{a}\ \vec{b}\ \vec{c}] \neq 0$ for $\vec{a}$, $\vec{b}$ and $\vec{c}$ are non-coplanar.